

Ultraviolet-Finite QFT on Curved Space-Times

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Abstract

This paper proposes a different interpretation on the renormalization program of an interacting Klein-Gordon field in curved space-times. Rather than being just another renormalization program we argue that it makes the QFT ultraviolet-finite at its foundation. More precisely, we argue that renormalization is a part of the foundation of QFT and that without it the theory is mathematically ill-defined or at the very least incompletely specified.

1 Introduction

“Most physicists are very satisfied with the situation. They say: ‘Quantum electrodynamics is a good theory, and we do not have to worry about it any more.’ I must say that I am very dissatisfied with the situation, because this so-called ‘good theory’ does involve neglecting infinities which appear in its equations, neglecting them in an arbitrary way. This is just not sensible mathematics. Sensible mathematics involves neglecting a quantity when it is small – not neglecting it just because it is infinitely great and you do not want it!” Dirac, source: “Dirac” By Helge Kragh 1990

In renormalization we subtract infinite constants and absorb infinite constants into the parameters. Any person who first sees this method is somewhat unsettled. Surely it works, because the predictions back it up, but we want to understand exactly why it works. In any physical theory we cannot simply “throw out” the infinities because we don’t like them.

The presence of infinities in it of itself is not a reason to disallow them because there is a reason that they are there. For instance, in GR we have singularity at the event horizon of a black hole solution provided by Schwarzschild. After much debate it is now accepted that this a defect of the chosen coordinates. The Kruskal extension of Schwarzschild coordinates is coordinate system that has no infinities at the event horizon. The event horizon, as we now know, is simply a region of space after which nothing can escape, not a singularity. However there is a more serious singularity (the point of infinity density) at the center of the black hole, but it is assumed that another theory, namely a unified theory or quantum gravity, takes hold at such densities and energies. Thus, infinities usually represent poor choices or limits of the theory and in the latter case their very presence means that some other theory must take over to properly describe the physics of these regions.

Renormalization is very different. When we try compute things that we should be able to compute, such as mass, they blow up to infinity. The only way, at the time it was formulated, to save the theory was to concoct a semi-mathematical method for absorbing or subtracting infinities. Some of the founders of QFT, such as Dirac, were very unsettled by these turn of events. In mathematics you cannot absorb or subtract infinities in this way. Despite all this renormalization works fantastically well when it comes to predictions in experiments. Thus, it is a consensus that renormalization could be (in the future) be put on firm mathematical ground and all can be justified, even while there maybe some iffy mathematics now in renormalization. This is the current viewpoint of most physicists today.

In Minkowski space QFT it is the interpretation taken by Scharf [2] that the Epstein-Glaser perturbative renormalization [1] is not merely another renormalization program – it makes QFT ultraviolet-finite. By improperly incorporating causality and splitting distributions into retarded and advanced parts, such as propagators, incorrectly you are punished by ultraviolet infinities. This program has been extended to curved space-time in [5]. To accomplish this two crucial requirements are needed: one is the proper inclusion of causality and the other is the micro-local spectrum

condition [5].¹ It is the author's opinion that Scharf's viewpoint holds in this case as well as we will argue.

Infinites in renormalization come two basic types: infrared and ultraviolet. The former regards more to questions on states such as the existence of vacuum states, thermal states, etc. (e.g. see [6]). The latter is a more serious problem from a foundational perspective, but in [5,3,4] a new paradigm for perturbative renormalization of interacting of QFT in curved space-times (QFTCS) has been defined to eliminate the ultraviolet infinities. The main insight is that by taking improper pointwise products of distributions you are punished by ultraviolet infinities. For example, the pointwise product of the Feynman propagator is *never* well-defined and so you would get ultraviolet infinities in that case. It is this new program that this paper focuses on.

With regards with renormalization, we have made major gains in understanding the sources of the infinities and where the current standard Minkowski space techniques go wrong. With respect to ultraviolet infinities, we could view this new correction of the infinities as another renormalization scheme of a QFTCS. However, the author proposes a different interpretation that based on the observation that the ultraviolet infinities arise from ill-defined products of distributions. Thus the "unrenormalized" theory is objectionable because these products are not well-defined in a mathematical sense. Thus, we could view the original "unrenormalized" theory as being not a viable candidate for a physical theory on these grounds. This then suggest that we should incorporate the definitions of products of distributions into the foundations of the theory. QFT becomes ultraviolet-finite since the renormalization program is "cooked" into it a priori.

In the sense above, the term renormalization is a misnomer. By saying this, the author does not advocate to get rid of the term – it's here to stay whether we like it or not. What the author simply wants is to rethink about the term itself and what previously meant and what it should mean now. Previously it meant that we start with the "unrenormalized theory" that is (as we believe it to be) correct as a physical theory, compute infinities everywhere, and get rid of them through renormalization to obtain the correct theory we actually use. In this new paradigm the "unrenormalized theory" cannot be regarded as even a candidate for a physical theory – because at the very least it is not specified well enough. Therefore, the author argues that renormalization is not only needed to obtain usable predictions, but justified both mathematically and physically in the specification of QFT. Renormalization should be viewed as a program to properly define the observable algebra in QFT.

To say that renormalization is misnomer is not to say that there are not any regularizations involved, but simply without it (including any regularization involved) QFT on curved space-time is simply mathematically incorrect or at the very least incomplete. The way to think of this interpretation is that it is a validation of what physicists already suspected: renormalization is correct and can be put on a solid mathematical foundation. This new interpretation of the renormalization program of QFTCS allows us to put it on a solid mathematical foundation and into the very definition of QFT.

2 An Interpretation of Perturbative Renormalization of QFTCS: Ultraviolet-Finite Perturbative QFT

In this section we define $\mathcal{W}(M, g)$ as the algebra of Wick polynomials, their time-order products, and derivatives constructed in [3,4]. We denote the constructions of the Wick powers and their time-ordered products presented in these references by the maps $N^k[\varphi] = \varphi^k$ and $T(\varphi^{k_1} \cdots \varphi^{k_n})$ respectively.

The maps N^k and T , the author claims, are fundamental to the specification of QFT. A possible issue arises because these maps have ambiguities in their definition, but this will not cause problems if the physical theory is independent of the choice. If this were the case then the physical predictions would be affected by the choice of maps N^k and T . Similar to choice of coordinates, physical theories must be independent of such artificial choices and we argue that this is the case here. The reasoning is based on the observation in [3] that the ambiguities of N^k and T are precisely the renormalization ambiguities.

Specifically they say the following. Let φ^k , $T(\prod_i \varphi^{k_i})$ and $N_H^k[\varphi] =: \varphi^k :_H$, $\tilde{T}(\prod_i \varphi^{k_i} :_H)$ be two different prescriptions for defining Wick powers and their time-ordered products. Let the S-matrix of our theory be

$$S(\mathcal{L}_{int}) = 1 + \sum_{n \geq 1} \frac{i^n}{n!} \int_{M^n} T(\mathcal{L}_{int}(x_1) \cdots \mathcal{L}_{int}(x_n) \mu_g(x_1) \cdots \mu_g(x_n))$$

¹The relation to Scharf's ideas are seen by the connection between the splitting of distributions (e.g. propagators) and the micro-local spectrum condition. The micro-local spectrum can be thought of as requiring that every propagator appearing in Feynman diagrams must be of Hadamard form.

where interaction part of the Lagrangian is $\mathcal{L}_{int}(x) = f\varphi^4(x)$ and the S-matrix in the other prescription is

$$\tilde{S}(\tilde{\mathcal{L}}_{int}) = 1 + \sum_{n \geq 1} \frac{i^n}{n!} \int_{M^n} \tilde{T}(\tilde{\mathcal{L}}_{int}(x_1) \cdots \tilde{\mathcal{L}}_{int}(x_n) \mu_g(x_1) \cdots \mu_g(x_n))$$

where $\tilde{\mathcal{L}}_{int}(x) = f : \varphi^4(x) :_H$. Note that f is a coupling function that is constant in some compact region of space (our interaction center).²

In [3] they showed that the ambiguities of the definitions of the Wick powers and their time-ordered products are precisely the renormalization ambiguities. I.e., that $\tilde{S}(\tilde{\mathcal{L}}_{int}) = S(\mathcal{L}_{int} + \delta\mathcal{L}_{int})$ for some local and covariant field $\delta\mathcal{L}_{int}$ which has the same form as the original Lagrangian $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}$. Thus the theories are equivalent and are related by a redefinition of the constants like the field strength and couplings (apart from terms proportional to the identity which contribute to the overall phase of S which are inconsequential for QFT). Thus, since the constants must be measured then the physical predictions will always be the same regardless of the choice of N^k and T . It is observed that:

- *A physical model which is renormalizable (i.e., their Lagrangians are of renormalizable form), such as Klein-Gordon with a φ^4 interaction term, together with all other axioms specified in [3,4] uniquely define the physical theory.*

Subsequently, suppose that we assume that renormalization is required for the specification of QFT *then only theories which are renormalizable can be candidates for the physical theories*. This should be viewed as starting point for QFT so that we disallow non-renormalizable theories and, therefore, inconsistent physical predictions.

The idea of ultraviolet-finite QFT is as follows:

Without any proper prescription for both N^k and T we do not have a mathematically well-defined theory, thus they are essential. This means that the “unrenormalized theory” cannot be even a possible candidate for a physical theory. As a result of ill-defined products of distributions we would be punished by ultraviolet infinities.

If we assume that the maps N^k and T are in the foundations of the theory then ambiguities are no problem only for Lagrangians of renormalizable form ($\delta\mathcal{L}_{int}$ which has the same form as the original Lagrangian). Then no matter what prescription you choose, say φ^k , $T(\prod_i \varphi^{k_i})$, the theory will be physically equivalent to all other choices since the form of the Lagrangian (and hence the S-matrix) is the same. Only the constants differ, but these are always fixed by experiments. In this way, the physics is completely determined regardless of prescription we choose. The “renormalized theory”, and not the “unrenormalized theory”, can be the only candidate for a physical theory by which we can make predictions.

We now can say the label renormalization is a misnomer, since infinities never arise when we compute any quantities that should be finite like mass or the coupling. This is seen by the fact that the renormalization scheme N^k and T is not an afterthought. We don’t compute quantities, such as mass, in what we assume is a perfectly well-defined theory, suddenly get infinities, and then have to rid ourselves of them somehow. The renormalization step is cooked into the very definition of the theory. *Renormalization allows us to define \mathcal{W}* which is the observable algebra. The ambiguities (which are finite) are not a problem (in terms of physical predictions) if we assume that the S-matrix is renormalizable since the form of the Lagrangian is the same. It is then only a matter of redefining our constants and then going out and use experiments to fix them.

3 Summary

The results of this paper does take an unconventional viewpoint, but the insights are based on quite conventional results of QFT on curved space-time. Namely that QFT is at the very least incompletely specified without renormalization, because operations with distributions (like products) must be carefully defined. That is renormalization is not only now mathematically well-defined, but it is “cooked” into the foundations of QFT – i.e. it is part of QFT’s definition. The author suggests that we should require that all candidates for physical theories to be ones which are renormalizable (i.e., their Lagrangians are of renormalizable form). This is to say that if a QFT makes physical and mathematical sense it must be renormalizable, and this should be a starting principle of QFT.

²This is an assumption of the renormalization program which allows prevent infrared infinities by truncating the S-matrix series. The limit of coupling functions to constants is the adiabatic limit and in this limit the infrared infinities must be dealt with.

If renormalization is a part of the definition of QFT then there are no ultraviolet infinities. This is not to say that there are no regularizations in the defining the Wick powers and their time-ordered products, there clearly are. These regularizations are not as a result of computing quantities that should be finite, but are not. Quite the contrary, it is the definition of the framework, i.e. the algebra of observables \mathscr{W} , that only uses these regularizations. So what the author proposes is that we interpret these regularizations as not only needed, but justified (both mathematically and physically) to be able to formulate QFT correctly. Having renormalization built into the theory becomes ultraviolet-finite.

4 References

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